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On the Biclique cover of the complete graph

Farokhlagha Moazami and Nasrin Soltankhah
Department of Mathematics
Alzahra University
Vanak Square 19834 Tehran, I.R. Iran
f.moazami@alzahra.ac.ir
soltan@alzahra.ac.ir

Abstract

Let K be a set of k positive integers. A biclique cover of type K of a graph G is a collection of complete bipartite subgraphs of G such that for every edge e of G, the number of bicliques need to cover e is a member of K. If $K = \{1, 2, \ldots, k\}$ then the maximum number of the vertices of a complete graph that admits a biclique cover of type K with d bicliques, $n(k, d)$, is the maximum possible cardinality of a k-neighborly family of standard boxes in $\mathbb{R}^d$. In this paper, we obtain an upper bound for $n(k, d)$. Also, we show that the upper bound can be improved in some special cases. Moreover, we show that the existence of the biclique cover of type K of the complete bipartite graph with a perfect matching removed is equivalent to the existence of a cross K-intersection family.

Key words: biclique cover- multilinear polynomial- cross K-intersection families.

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1 Introduction

Throughout the paper, we consider only the simple graph. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of the graph G. As usual we will use the symbol $[n]$ to denote the set $\{1, 2, \ldots, n\}$. By a *biclique* we mean the complete bipartite graph. We will denote by $K^-_{m,m}$ the complete bipartite graph with a perfect matching removed. A *biclique cover* of a graph G is a collection of bicliques of G such that each edge of G belongs to at least one of the bicliques. In the literature, there are several ways to define a biclique cover problem for different purposes, see [2, 3, 4, 5, 7, 8, 9, 10, 14, 13]. In this paper we consider the biclique cover of order k and in general of type K that was introduced by Alon [2].

Definition 1. Let K be a set of k positive integers. We say that a biclique cover of the graph G is of type K if for every edge e of the graph G, the number of bicliques that cover e is an element of the set K. ♠

The number of bicliques in a biclique cover is the *size* of the cover. We write $n(K, d)$ for the maximum number of vertices of a complete graph that admits a biclique cover of type K and size d. In the aforementioned definition if we take the set $K = \{1, 2, \ldots, k\}$ then this biclique cover is called of *order* k. Alon [2] used the concept of the biclique cover and provided a relation between the biclique cover of

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order k and size d of the complete graph and k-neighborly family of standard boxes in $\mathbb{R}^d$, (see [2] for more details.) The following result is due to N. Alon [2].

Theorem A. [2] *Let d be a positive integer and $1 \leq k \leq d$ then*

1. $d+1 = n(1,d) \leq n(2,d) \leq \cdots \leq n(d-1,d) \leq n(d,d) = 2^d$.
2. $(\frac{d}{k})^k \leq \prod_{i=0}^{k-1}(\lfloor \frac{d+i}{k} \rfloor + 1) \leq n(K,d) \leq \sum_{i=0}^{k} 2^i \binom{d}{i} < 2(\frac{2ed}{k})^k$.

To prove the upper bound in the second relation he constructed multilinear polynomials that are linearly independent. The proof can be easily extended to the biclique cover of type K. In this paper, we add some other linearly independent multilinear polynomials and obtain a slightly improvement bound for the mentioned upper bound.

Theorem 1. *For $1 \leq k \leq d$, assume that K is a set of k positive integers. Then*

$$n(K,d) \leq 2^k \binom{d}{k} + \sum_{i=1}^{k-1} 2^i \binom{d-1}{i-1}.$$

Also, it was shown by Alon [2] that for $K = \{2, 4, \ldots, 2i\}$ there exists a biclique cover $H_1, H_2, \ldots, H_d$ of type K for the complete graph on $(1 + \binom{d}{2} + \binom{d}{4} + \cdots + \binom{d}{2i})$ vertices which improves the lower bound in Theorem A. To see this, assume that the vertices of the complete graph is denoted by all subsets of cardinality $0, 2, 4, \ldots, 2i$ of $[d]$. Construct the biclique cover $H_1, H_2, \ldots, H_d$ which H_i has (X_i, Y_i) as vertex classes such that X_i is the set of all subsets that contain i and Y_i is the set of all subsets that do not contain i. These results are far from being optimal, but the above construction motivated us to define the following definition.

Definition 2. Suppose that $\{G_1, \ldots, G_d\}$ is a biclique cover of the graph G where G_i has (X_i, Y_i) as its vertex set. If for every $1 \leq i \leq d$, $X_i \cup Y_i = V(G)$, then this biclique cover is called a *regular biclique cover.* ♠

We will denote by $n_r(K,d)$ the maximum possible cardinality of the vertices of a complete graph such that there exists a regular biclique cover of type K and size d. We prove the following upper bound of $n_r(K,d)$.

Theorem 2. *If K is a set of k positive integers, then*

$$n_r(K,d) \leq 2^{k-1}\binom{d}{k} + 2^{k-1}\binom{d-1}{k} + 2^k - 1.$$

It is interesting to consider the biclique cover of type K of other graphs besides the complete graph. Assume that K is a set of k positive integers and X is an arbitrary set of d points. Suppose that $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ are two collections of subsets of X such that $|A_i \cap B_j| \in K$ for $i \neq j$ and $|A_i \cap B_i| = 0$ for every i. The pair $(\mathcal{A}, \mathcal{B})$ is called a cross K-intersection families. The following theorem shows that a cross K-intersection family can be formulated in terms of biclique cover of type K of the graph $K^-_{m,m}$.

Theorem 3. *Let $K = \{l_1, l_2, \ldots, l_k\}$, there exists a cross K-intersection families with m blocks on a set of d points if and only if there exists a biclique cover of type K and size d of $K^-_{m,m}$.*

In [12] Snevily made the following conjecture.

Conjecture 1. [12] *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ be two collections of subsets of an d-element set. Let $K = \{l_1, l_2, \ldots, l_k\}$ be a collection of k positive integers. Assume that for $i \neq j$ we have $|A_i \cap B_j| \in K$ and that $|A_i \cap B_i| = 0$, then*

$$m \leq \binom{d}{k}.$$

By Theorem 3, we can state the above conjecture in terms of the biclique cover as follows. The maximum number of the vertices in each part of a complete bipartite graph with a perfect matching removed that admits a biclique cover of size d is at most $\binom{d}{k}$. Note that this bound is sharp by taking all k-element subsets of $[d]$ as $\mathcal{A}$ and all $(d-k)$-element subsets of $[d]$ as $\mathcal{B}$. In [6] William Y.C. Chen and Jiuqiang Liu proved the following theorem.

Theorem B. [6] *Let p be a prime number and $K = \{l_1, l_2, \ldots, l_k\} \subseteq \{1, 2, \ldots, p-1\}$. Assume that $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ are two collections of subsets of X such that $|A_i \cap B_j| (modp) \in K$ for $i \neq j$ and $|A_i \cap B_i| = 0$ for every i. If $maxl_j < min\{|A_i| (modp) | 1 \leq i \leq m\}$, then*

$$m \leq \binom{d-1}{k} + \binom{d-1}{k-1} + \cdots + \binom{d-1}{k-2r+1},$$

where r is the number of different set sizes in $\mathcal{A}$.

Clearly, for a prime number p greater than d, and $r = 1$ the above theorem is true and the following corollary is straightforward.

Corollary 1. [6] *Let $K = \{l_1, l_2, \ldots, l_k\}$ be a set of k positive integers and $\max\, l_j < s$. Suppose that $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ are two collections of subsets of $[d]$ such that $|A_i \cap B_j| \in K$ for $i \neq j$ and $|A_i \cap B_i| = 0$ for every i. If either $\mathcal{A}$ is s-uniform or $\mathcal{B}$ is s-uniform, then*

$$m \leq \binom{d}{k}.$$

This corollary shows that if we have a biclique cover of type $K = \{l_1, \ldots, l_k\}$ of a complete bipartite graph with a perfect matching removed such that every vertex of this graph lies in exactly s bicliques and $max\ l_j < s$. Then the maximum possible cardinality of the vertices of this graph is at most $\binom{d}{k}$. The structure of the rest of this paper is to prove Theorem 1, 2, and 3. The proofs based on the concept of applying linear algebra method that is used in [1, 6, 11, 12].

2 Proof of Theorem 1

A polynomial in n variable is called *multilinear* if every variable has degree 0 or 1. Observe that when each variable in a polynomial attains values 0 or 1, if each variable x_i^p $(p > 1)$ is replaced by x_i, we can consider this polynomial as a multilinear polynomial. For a subset A_i of $[n]$, the *characteristic vector* of A_i is the vector $v_{A_i} = (v_1, \dots, v_n)$, where $v_j = 1$ if $j \in A_i$ and $v_j = 0$ otherwise. Let $\{H_1, H_2, \dots, H_d\}$ be a biclique cover of type K for the graph K_n such that H_i has X_i and Y_i as its vertex classes. For every $1 \le i \le n$ define

$$A_i := \{j \mid i \in X_j\} \quad \& \quad B_i := \{j \mid i \in Y_j\}.$$

Now, with each pair (A_i, B_i) we associate a polynomial $P_i(x,y)$ defined by:

$$P_i(x,y) = \prod_{l_j \in K} (v_{A_i}.x + v_{B_i}.y - l_j).$$

Where v_{A_i} (resp. v_{B_i}) is the characteristic vector of the set A_i (resp. B_i), $x = (x_1, x_2, \dots, x_d)$ and $y = (y_1, y_2, \dots, y_d)$. The key property of these polynomials is

$$P_i(v_{B_i}, v_{A_i}) \neq 0 \text{ and } P_i(v_{B_j}, v_{A_j}) = 0 \text{ for all } i \neq j, \tag{1}$$

which follows immediately from this fact that $\{H_1, \dots, H_d\}$ is a biclique cover of type K. Let

$$\mathcal{A} = \{(M,N) \mid M, N \subseteq [d],\ M \cap N = \emptyset,\ d \in N,\ |M \cup N| \le k\}.$$

It is easy to see that the cardinality of $\mathcal{A}$ is equal to $\sum_{i=0}^{k-1} 2^i \binom{d-1}{i}$. For every $(M,N) \in \mathcal{A}$, the polynomial $Q_{(M,N)}(x,y)$ is defined by

$$Q_{(M,N)}(x,y) = \prod_{i \in M} x_i \prod_{i \in N} y_i.$$

Throughout the paper we set $\prod_{i \in A} x_i = \prod_{i \in A} y_i = 1$ when A is an empty set. We will now show that the polynomials in the set

$$\mathcal{P} = \{P_i(x,y) \mid 1 \le i \le n\} \cup \{Q_{(M,N)}(x,y) \mid (M,N) \in \mathcal{A}\}$$

, as the polynomials from $\{0,1\}^{2d}$ to $\mathbb{R}$, are linearly independent. For this purpose, we set

$$\sum_{i=1}^{n} \alpha_i P_i(x,y) + \sum_{(M,N) \in \mathcal{A}} \beta_{(M,N)} Q_{(M,N)}(x,y) = 0.$$

We can rewrite the above equality as follows:

$$\sum_{d \in A_i} \alpha_i P_i(x,y) + \sum_{d \notin A_i} \alpha_i P_i(x,y) + \sum_{(M,N) \in \mathcal{A}} \beta_{(M,N)} Q_{(M,N)}(x,y) = 0. \tag{2}$$

The proof will be divided into 3 steps.

Step 1. We begin by proving that for every i which $d \notin A_i$, it holds that $\alpha_i = 0$. In the contrary assume that i_0 is a subscript such that $\alpha_{i_0} \neq 0$ and $d \notin A_{i_0}$. Substituting $(v_{B_{i_0}}, v_{A_{i_0}})$ in the equation 2, according to the relation 1 and since $d \in N$, all terms in the relation 2 but $\alpha_{i_0} P_{i_0}(v_{B_{i_0}}, v_{A_{i_0}})$ vanish. In this way, $\alpha_{i_0} P_{i_0}(v_{B_{i_0}}, v_{A_{i_0}}) = 0$. Finally, as $P_{i_0}(v_{B_{i_0}}, v_{A_{i_0}}) \neq 0$, we have $\alpha_{i_0} = 0$ which is a contradiction.

Step 2. We will show that for every i which $d \in A_i$, it holds that $\alpha_i = 0$. According to the step 1 we have

$$\sum_{d \in A_i} \alpha_i P_i(x, y) + \sum_{(M,N) \in \mathcal{A}} \beta_{(M,N)} Q_{(M,N)}(x, y) = 0. \tag{3}$$

Assume that i_0 is a subscript such that $\alpha_{i_0} \neq 0$. Let $v'_{A_{i_0}} = v_{A_{i_0}} - (0, \ldots, 0, 1)$ and evaluate the equation 3 in $(v_{B_{i_0}}, v'_{A_{i_0}})$. As $A_i \cap B_i = \emptyset$ we have that $d \notin B_i$ so for every i in the equation 3 we have $P_i(v_{B_{i_0}}, v'_{A_{i_0}}) = P_i(v_{B_{i_0}}, v_{A_{i_0}})$. From this and since $d \in N$ we conclude that $\alpha_{i_0} P_{i_0}(v_{B_{i_0}}, v_{A_{i_0}}) = 0$, hence $\alpha_{i_0} = 0$.

Step 3. Note that $Q_{(M',N')}(v_{M'}, v_{N'}) = 1$ and $Q_{(M,N)}(v_{M'}, v_{N'}) = 0$ for any $(M, N) \in \mathcal{A}$ with $M \cup N \neq M' \cup N'$ and $|M \cup N| \geq |M' \cup N'|$. So all the polynomials of the set $\{Q_{(M,N)}(x, y) \mid (M, N) \in \mathcal{A}\}$ are linearly independent. Therefore, for every $(M, N) \in \mathcal{A}$ we have $\beta_{(M,N)} = 0$.

We have thus proved the polynomials of the set $\mathcal{P}$ as the polynomials with domain $\{0, 1\}^{2d}$ are linearly independent. So we can consider these polynomials as multilinear polynomials. On the other hand every polynomial in the set $\mathcal{P}$ can be written as a linear combination of the multilinear monomials of degree at most k. Furthermore, they do not contain any monomials that contain both x_i and y_i for the same i. The number of such monomials are $\sum_{i=0}^{k} 2^i \binom{d}{i}$ and hence,

$$n + \sum_{i=0}^{k-1} 2^i \binom{d-1}{i} \leq \sum_{i=0}^{k} 2^i \binom{d}{i}.$$

Now, by a straightforward calculation the formula of Theorem 1 will be achieved.

3 Proof of Theorem 2

Before embarking on the proof of Theorem 2, we will establish the following lemma.

Lemma 1. *For every $1 \leq i \leq d-1$, let the set $\mathcal{B}_i$ define as follows:*

$$\mathcal{B}_i = \{(I, J) | I, J \subseteq [d-i+1],\ I \cap J = \emptyset,\ d-i+1 \in I \cup J,\ |I \cup J| \neq d-i+1,\ |I \cup J| \leq k-1\}.$$

Let for every pair $(I, J) \in \mathcal{B}_i$,$R^i_{(I,J)}(x, y)$ denote the following polynomial

$$R^i_{(I,J)}(x, y) = \prod_{j \in I} x_j \prod_{j \in J} y_j \Big(\sum_{j \notin J, j \leq d-i} x_j + \sum_{j \notin I, j \leq d-i} y_j - (d-i) \Big).$$

Then

$$\mathcal{B} = \{R^i_{(I,J)}(x,y) \mid 1 \le i \le d-1,\ \ (I,J) \in \mathcal{B}_i\} \tag{4}$$

is a set of linearly independent polynomials.

Proof. To prove the assertion, assume this is false and let

$$\sum_{(I,J)\in\mathcal{B}_1} \gamma^1_{(I,J)} R^1_{(I,J)}(x,y) + \cdots + \sum_{(I,J)\in\mathcal{B}_{d-1}} \gamma^{d-1}_{(I,J)} R^{d-1}_{(I,J)}(x,y) = 0 \tag{5}$$

be a nontrivial linear relation. Suppose that i_0 is the greatest superscript and (I_0, J_0) is the subscript such that has minimum cardinality in the set $\mathcal{B}_{i_0}$ and $\gamma^{i_0}_{(I_0,J_0)} \neq 0$. Substitute (v_{I_0}, v_{J_0}) for (x,y) in the linear relation 5. In view of the definition of $\mathcal{B}_i$ all terms in the linear relation 5 but $\gamma^{i_0}_{(I_0,J_0)} R^{i_0}_{(I_0,J_0)}(v_{I_0}, v_{J_0})$ vanish. Since $R^{i_0}_{(I_0,J_0)}(v_{I_0}, v_{J_0}) \neq 0$, we have $\gamma^{i_0}_{(I_0,J_0)} = 0$. This is a contradiction which completes the proof. ■

Obviously, for every $1 \le i < d$ we have

$$|\mathcal{B}_i| = \begin{cases} \sum_{j=0}^{k-2} 2^{j+1}\binom{d-i}{j} & i \le d-k+1 \\ \\ \sum_{j=0}^{d-i-1} 2^{j+1}\binom{d-i}{j} & i \ge d-k+2 \end{cases}.$$

It is a well-known fact that

$$\sum_{i=0}^{n} \binom{m+i}{m} = \binom{m+n+1}{m+1}.$$

By this fact clearly, $|\mathcal{B}| = \sum_{j=1}^{k-1} 2^j \binom{d}{j} - 2^k + 2$. Let $\mathcal{A}_k = \{(M,N) \mid (M,N) \in \mathcal{A},\ |M \cup N| = k\}$ and $\mathcal{B}$ is defined as Lemma 1. We claim that $\{P_i(x,y) \mid 1 \le i \le n\} \cup \{Q_{(M,N)} \mid (M,N) \in \mathcal{A}_k\}$ with all the polynomials $R^i_{(I,J)}(x,y) \in \mathcal{B}$ remain linearly independent. Before prove the claim, we shall note that all polynomials in the set $\mathcal{B}$ have this property that vanish in the point (v_{B_i}, v_{A_i}) for every $1 \le i \le n$. Now, assume the claim is false and let

$$\sum_{i=1}^{n} \alpha_i P_i(x,y) + \sum_{(M,N)\in\mathcal{A}_k} \beta_{(M,N)} Q_{(M,N)}(x,y) + \sum_{i=1}^{d-1} \sum_{(I,J)\in\mathcal{B}_i} \gamma^i_{(I,J)} R^i_{(I,J)}(x,y) = 0 \tag{6}$$

be a nontrivial linear relation.

Step 1. Let i_0 be a subscript such that $d \notin A_{i_0}$ and $\alpha_{i_0} \neq 0$. Substitute $(v_{B_{i_0}}, v_{A_{i_0}})$ for (x,y) in the linear relation 6. We know that for every i and every $(I,J) \in \mathcal{B}_i$, $R^i_{(I,J)}(v_{B_{i_0}}, v_{A_{i_0}}) = 0$. Also, $d \in N$ so $Q_{(M,N)}(v_{B_{i_0}}, v_{A_{i_0}}) = 0$. Using these and by 1 all terms in the linear relation 6 but $\alpha_{i_0} P_{i_0}(v_{B_{i_0}}, v_{A_{i_0}})$ vanish. Since $P_{i_0}(v_{B_{i_0}}, v_{A_{i_0}}) \neq 0$, therefore $\alpha_{i_0} = 0$.

Step 2. According to the step 1 we have

$$\sum_{d\in A_i}^{n} \alpha_i P_i(x,y) + \sum_{(M,N)\in\mathcal{A}_k} \beta_{(M,N)} Q_{(M,N)}(x,y) + \sum_{i=1}^{d-1} \sum_{(I,J)\in\mathcal{B}_i} \gamma^i_{(I,J)} R^i_{(I,J)}(x,y) = 0. \tag{7}$$

Let i_0 be a subscript such that $\alpha_{i_0} \neq 0$. We define $v'_{A_{i_0}}$ to be $v_{A_{i_0}} - (0,\ldots,0,1)$. Substitute $(v_{B_{i_0}}, v'_{A_{i_0}})$ for (x,y) in the linear relation 7. For $1 \leq i \leq d-1$ and every pair $(I,J) \in \mathcal{B}_i$, by the definition of $R^i_{(I,J)}(x,y)$, it holds that $R^i_{(I,J)}(v_{B_j}, v_{A_j}) = R^i_{(I,J)}(v_{B_j}, v'_{A_j})$ for every $1 \leq j \leq n$. Now, similar in the step 1 all terms in the relation 7 but $\alpha_{i_0} P_{i_0}(v_{B_{i_0}}, v'_{A_{i_0}})$ vanish. Since $P_{i_0}(v_{B_{i_0}}, v'_{A_{i_0}}) \neq 0$, therefore $\alpha_{i_0} = 0$. So, we have

$$\sum_{(M,N)\in\mathcal{A}_k} \beta_{(M,N)} Q_{(M,N)}(x,y) + \sum_{i=1}^{d-1} \sum_{(I,J)\in\mathcal{B}_i} \gamma^i_{(I,J)} R^i_{(I,J)}(x,y) = 0. \tag{8}$$

Step 3. Since $d \in N$ and $d \notin J$ for all $(I,J) \in \mathcal{B}_i$, $i = 2,\ldots,d-1$, if we evaluate equality 8 in (v_I, v_J) then we conclude that $\gamma_{(I,J)} = 0$. Hence we have

$$\sum_{(M,N)\in\mathcal{A}_k} \beta_{(M,N)} Q_{(M,N)}(x,y) + \sum_{(I,J)\in\mathcal{B}_1} \gamma^1_{(I,J)} R^1_{(I,J)}(x,y) = 0. \tag{9}$$

Assume that (I_0, J_0) is a subscript such that has minimum cardinality in the set $\mathcal{A}_1$ such that $\gamma_{(I_0,J_0)} \neq 0$. Substituting (v_{I_0}, v_{J_0}) in the equation 9 then since $|I_0 \cup J_0| \leq k-1$ and $|M \cup N| = k$ all terms in 9 but $\gamma_{(I_0,J_0)} R^1_{(I_0,J_0)}(v_{I_0}, v_{J_0})$ vanish. So we have $\gamma_{(I_0,J_0)} = 0$

Step 4. By independence of the polynomials in the set $\mathcal{P}$, each $\beta_{(M,N)} = 0$, therefore the claim is true.

So, we have $n + 2^{k-1}\binom{d-1}{k-1} + \sum_{i=1}^{k-1} 2^i \binom{d}{i} - 2^k + 2$ linearly independent polynomials which, as the proof of Theorem 1, are in the space generated by $\sum_{i=0}^{k} 2^i \binom{d}{i}$ monomials. Hence,

$$n(K,d) \leq 2^k \binom{d}{k} - 2^{k-1}\binom{d-1}{k-1} + 2^k - 1.$$

That complete the proof of Theorem 2.

4 Proof of Theorem 3

Let $K = \{l_1, l_2, \ldots, l_k\}$, and $(\mathcal{A}, \mathcal{B})$ be a cross K-intersection families with m blocks on a set of d points. For every $1 \leq j \leq d$, let

$$X_j \stackrel{\text{def}}{=} \{i \mid 1 \leq i \leq m, j \in A_i\},$$

$$Y_j \stackrel{\text{def}}{=} \{i|\ 1 \le i \le m, j \in B_i\}.$$

Now, for $j = 1, 2, \dots, d$, we construct the complete bipartite graph G_j with vertex set (X_j, Y_j), where X_j and Y_j were defined as above. Let ij be an arbitrary edge of $K^-_{m,m}$, consider sets A_i and B_j. Without loss of generality assume that $A_i \cap B_j = \{v_1, v_2, \dots, v_l\}$, which $1 \le l \le k$. It is not difficult to see that the edge ij was covered by the graphs $G_{v_1}, G_{v_2}, \dots, G_{v_l}$. So we have a biclique cover of type K and size d of $K^-_{m,m}$. Conversely let $\{G_1, G_2, \dots, G_d\}$ be a biclique cover of type K and size d of the graph $K^-_{m,m}$. Assume G_i has (X_i, Y_i) as the vertex set. For every $1 \le j \le m$ define

$$A_j \stackrel{\text{def}}{=} \{i|\ 1 \le i \le d, j \in X_i\},$$

$$B_j \stackrel{\text{def}}{=} \{i|\ 1 \le i \le d, j \in Y_i\}.$$

Let $\mathcal{A} = \{A_1, \dots, A_m\}$ and $\mathcal{B} = \{B_1, \dots, B_m\}$. Since $X_i \cap Y_i = \varnothing$ for every $1 \le i \le d$ then $A_j \cap B_j = \varnothing$ for every $1 \le j \le m$. Also, if $\{G_{v_1}, G_{v_2}, \dots, G_{v_l}\}$ is the set of graphs that cover the edge ij then $A_i \cap B_j = \{v_1, v_2, \dots, v_l\}$ where $|A_i \cap B_j| \in K$. Hence $(\mathcal{A}, \mathcal{B})$ is a cross K-intersection family.

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References

[1] N. Alon, L. Babai, and H. Suzuki. Multilinear polynomials and Frankl–Ray-Chaudhuri–Wilson type intersection theorems. *J. Combin. Theory Ser. A*, 58(2):165–180, 1991.

[2] Noga Alon. Neighborly families of boxes and bipartite coverings. In *The mathematics of Paul Erdős, II*, volume 14 of *Algorithms Combin.*, pages 27–31. Springer, Berlin, 1997.

[3] S. Bezrukov, D. Fronček, S. J. Rosenberg, and P. Kovář. On biclique coverings. *Discrete Math.*, 308(2-3):319–323, 2008

[4] Béla Bollobás and Alex Scott. On separating systems. *European J. Combin.*, 28(4):1068–1071, 2007.

[5] Béla Bollobás and Alex Scott. Separating systems and oriented graphs of diameter two. *J. Combin. Theory Ser. B*, 97(2):193–203, 2007.

[6] William Y. C. Chen and Jiuqiang Liu. Set systems with l-intersections modulo a prime number. *J. Combin. Theory Ser. A*, 116(1):120–131, 2009.

[7] P. Erdős and L. Pyber. Covering a graph by complete bipartite graphs. *Discrete Math.*, 170(1-3):249–251, 1997.

[8] H. Hajiabolhassan and F. Moazami. Secure frameproof code through biclique cover. Available online at http://arxiv.org/abs/1202.1889v1.

[9] H. Hajiabolhassan and F. Moazami. Some new bounds for cover-free families through biclique cover. Available online at http://arxiv.org/abs/1008.3691v2.

[10] S. Jukna. On set intersection representations of graphs. *Journal of Graph Theory*, 61(1):55–75, 2009.

[11] Jian Liu and Jiuqiang Liu. Set systems with cross L-intersection and k-wise L-intersecting families. *Discrete Math.*, 309(20):5920–5925, 2009.

[12] Hunter S. Snevily. A sharp bound for the number of sets that pairwise intersect at k positive values. *Combinatorica*, 23(3):527–533, 2003.

[13] V. L. Watts. Fractional biclique covers and partitions of graphs. *Electron. J. Combin.*, 13(1):Research Paper 74, 14 pp. (electronic), 2006.

[14] V. L. Watts. *Covers and partitions of graphs by complete bipartite subgraphs*. ProQuest LLC, Ann Arbor, MI, 2001. Thesis (Ph.D.)–Queen's University (Canada).